CLASS24

Excellent Mathematics – 2
Target: NMTC Final Round
Sub Junior (Class-VII/VIII)

CONTENTS

CLASS24

EXCELLENT MATHEMATICS-2

Target : NMTC Final Round

CHAPTER 1 Basic Number Operations

1. A book with 240 pages is to have its pages numbered in the usual fashion, how many digits will this need ?
2. A book has pages numbered 1 to 192 (totally 96 sheets). Some 25 sheets are pulled out of it at random; show that the sum of these 50 numbers cannot be equal to 2002.
3. Find the number of two digit numbers whose sum of the digits is a single digit number.
4. Find the sum of all three digit numbers that can be written using the digits 1, 2, 3, 4, (repetitions allowed).
5. The natural numbers from 1 to 2100 are entered sequentially in 11 columns, with the first 3 rows as shown.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11
Row 1	1	2	3	4	5	6	7	8	9	10	11
Row 2	12	13	14	15	16	17	18	19	20	21	22
Row 3	23	24	25	26	27	28	29	30	31	32	33
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

If the number 2002 occurs in column p and row q, find the value of p + q.

6. In the given addition, each letter stands for a natural number. (Identical letters denote the same number). Find the number for each letter.

$$\begin{array}{r} O\ N\ E \\ +\ F\ O\ U\ R \\ \hline F\ I\ V\ E \\ \hline \end{array}$$

7. Observe the following addition problem :

$$\begin{array}{r} A\ B\ C \\ +\ A\ B\ C \\ +\ A\ B\ C \\ \hline C\ C\ C \\ \hline \end{array}$$

Each letter stands for a particular number. Find the numbers and rewrite the problem using numbers.

8. Find the sum of all the digits of the result of the subtraction $10^{99} - 99$.
9. This year my age is a multiple of 7. Next year it will be a multiple of 5. I am above 20 years old but less than 80. What is my age ?
10. What is the smallest number n greater than 1 such that $\sqrt{1+2+3+....+n}$ is a positive integer?
11. A six-digit number of the form ABCABC is always divisible by 7 or 11 or 13. Explain why.
12. Given 72x = A679B where A, B are unknown whole-number digits, find X.
13. Using only the digits 0, 1, 2, 3, 5 (with no repetitions), three-digit numbers are formed. How many of them are multiples of 6 ?

14. Find the quotient of the least common multiple of the first 40 natural numbers divided by the least common multiple of the first 30 natural numbers.

15. Find a least integer n that leaves remainders 2, 3, 4 when divided by 3, 4, and 5 respectively.

16. The greatest common divisor of a and 72 is (a, 72) = 24 and the least common multiple of b and 24 is [b, 24] = 72. Find the g.c.d (a, b) and the ℓ c.m. [a, b] given that a is the smallest three digit number having this property; and b is the biggest integer having this property.

17. The sum and least common multiple of two positive integers x, y are given as x + y = 40 and 1 c.m. [x, y] = 48. find the numbers x and y.

18. What is the greatest positive integer n which makes $n^3 + 100$ divisible by n + 10 ?

19. The ten digit number 3A55B1063C is a multiple of 792. Find A, B, C.

20. Rekha was asked to add 14 to a certain number and then divide the result by 4. Intstead she first added 4 and then divided by 14. Her result was 5. Had she followed the instructuions correctly, by how much would her result have differed from the incorrect result ?

21. In the following display each letter represents a digit | 3 | B | C | D | E | 8 | G | H | I |. If the sum of any three successive digits is 18, find the value of H.

22. The ratio between a two-digit number and the sum of the digits of that number is a : b. If the digits in the units palce is n more than the digit in the tens place, prove that the number is given by $\frac{9na}{11b-2a}$.

23. (a) A two digit number is equal to six times the sum of its digits. Prove that the two digit number formed by interchanging the digits is equal to five times sum of its digits.

(b) Show that $\frac{10^{2013}+1}{10^{2014}+1} > \frac{10^{3013}+1}{10^{3014}+1}$

24. (a) For any two natural numbers m, n prove that $(m^3 + n^3 + 4)$ cannot be a perfect cube.

(b) A circle is divided into six sectors and the six numbers 1, 0, 1, 0, 0, 0 are written clockwise, one in each sector. One can add 1 to the numbers in any two adjacent sectors. Is it possible to make all the numbers equal? If so after how many operations can this be achieved?

25. (a) All natural numbers from 1 to 2013 are written in a row in that order. Can you insert + and – signs between them so that the value of the resulting expression is zero? If it is possible, how many + and – signs should be inserted? Justify your answer by giving clear reasoning.

(b) The natural number 1, 2, 3, ... are partitioned into subsets $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$ and so on. What are the greatest and least numbers in the set S_{2013}?

26. (a) The diagram below contains 13 boxes. The numbers in the second and twelth boxes are respectively 175 and 70. Fill up the boxes with natural numbers such that

i. sum of all numbers in the 13 boxes is 2015

ii. sum of the numbers in any three consecutives boxes is always the same

	175										70	

(b) If x,y, y, z are real and unequal numbers, prove that $2015x^2 + 2015y^2 + 6z^2 > 2(2012xy + 3yz + 3zx)$

27. A man is walking from a town A to another town B at a speed of 4 kms per hour. A bus started from town A one hour later and is travelling at a speed of 12 kms per hour. The man on the way got into the bus when it reached him and travelled further two hours to reach the town B. What is the distance between the towns A and B?

28. Let m, n, p be distinct two digit natural numbers. If $m = 10a + b$, $n = 10b + c$, $p = 10c + a$ find all possible values of gcd(m, n, p).

ANSWERS

1. 1332 digits **3.** 45 **4.** 17760 **5.** 193

6. There are many answers. Two of them are :

$$\begin{array}{r} 342 \\ +1350 \\ \hline 1692 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 462 \\ +8450 \\ \hline 8912 \\ \hline \end{array}$$

7. C = 5, B = 8, A = 1 **8.** 874 **9.** 49 years **10.** 8 **12.** 511

13. 7 possibilities (120, 210, 150, 510, 102, 132, 312) **14.** 2294 **15.** 59

16. g.c.d (a,b) = g.c.d. (120,72) = 24 and l.c.m (a,b) = l.c.m. (120,72) = 360

17. x = 24, y = 16 or x = 16, y = 24 **18.** 890 **19.** A = 4, B = 7, C = 2

20. 15 less than the correct answer. **21.** 7

24. (b) Not possible, cann't achieved.

25. (a) Result cann't be zero, (b) 2027091, 2025079

26. (a) 207, 175, 70, 207, 175, 70, 207, 175, 70, 207, 175, 70, 207

27. 30 kms

28. (11, 12, 21), (22, 24, 42), (33, 39, 93), (44, 48, 84), (14, 49, 91), (13, 39, 91); (28, 84, 42)

CHAPTER 2 Ratio and Percentage

1. From a group of boys and girls, 15 girls leave first. Then the ratio of the number of girls to the number of boys becomes 1 : 2. After this, 40 boys leave the group. Now the boys and girls are equal in number. How many girls were there in the beginning ?
2. Four person A, B, C, D went to a hotel to take tiffin. the total bill was Rs.60.
   A agreed to pay half the sum of the amounts paid by the other three;
   B agreed to pay one-third of the sum of the amounts paid by the others three;
   C agreed to pay one-fourth of the sum of the amounts paid by the other three;
   How many did D pay ?
3. The sizes of copier paper have the property that a sheet of paper cut in half gives two smaller sheets of the same shape as the original sheet. find the ratios of the sides of the sheets.
4. Place one non-zero digit in each box given below in such a way that the resulting equation valid :
   □ □ % of □ □ □= 400
5. When the price of a gas cylinder is increased by 20%, by what percent should a householder reduce his consumption such that there is no increase in his expenditure ?
6. The length of the base of a rectangle is increased by 10% but the area remains unchanged. By what percent was the breadth reduced ?
7. There was four tests in Mathematics. A student scored 84, 78 and 76 in the first three tests, each out of 100. How many marks should she score out of 200 in the fourth test so that the average on all the four tests would be 90% ?
8. The sum of the ages of a man and his wife is six times the sum of the ages of their children. Two years ago the sum of their ages was ten times the sum of the ages of their children. Six years hence the sum of their ages will by three times the sum of the ages of their children. How many children do they have?

ANSWERS

1. 55 girls **2.** Rs. 13 **3.** $\sqrt{2}:1$ **4.** 64% of 625 **5.** $16\frac{2}{3}\%$ **6.** $9\frac{1}{11}\%$

7. Not possible **8.** 3

CHAPTER 3 Algebra

1. A question paper has n questions ; (n > 20). Out of the first 20, a student answer 15 correctly and out of the remaining he answers one-third correclty. If all questions carry equal marks, and the student's total score is 50%, find the number of questions in the paper.

2. There is a group of cows and chickens. The number of legs was 14 more than twice the number of heads. Find the number of cows and chicknes.

3. The sum of four positive numbers is 680. If 5 is added to the first number, 5 is subtracted from the second, the third is multiplied by 5 and the positive square root of the fourth is extracted, we then get four equal numbers. What are the initial four numbers ?

4. Find all the two-digit numbers such that when they are divided by the sum of their digits, the quotient is 7 with no remainder.

5. Two candles A and B of the same height are lighted at the same instant. A is consumed in 4 hours while B in 3 hours. Assume each candle burns at a constant rate. In how many hours after being lighted was A twice the height of B ?

6. A man is due at a certain place at a certain time. If he walks at the rate of 6 kms per hour, he will be 15 minutes late; if he walks at the rate of 8 kms an hour, he will be 15 minutes early. Find the distance he has to walk.

7. A man loses one-third of his money; then he gains Rs.10. Then he loses one-third of his possession and again gain Rs.20. Now he find that he has what exactly he had at the beginning. What was the amount he had originally

8. For a class, copies of 9 maths books and 16 science books cost Rs.220. Each books costs a whole number of rupees. Find the cost of each maths book.

9. Write down your age in years. Multiply the number you have written by 10 and add 5. Multiply this sum by 10 again. Add the number of month in which you were born counting January as 1, February as 2 and so on. Subtract 50. The first two numerals on the left will be your age. The next two will be the number of month in which you were born. Explain how this works.

10. On September 1, 2002, a mason was appointed by a contractor at Rs.150 per day, subject to the condition that whenever the mason was absent he would be fined at Rs.200 per day, as penalty. At the end of the month the mason found that he earned Rs.650 only. How many days did he work for ?

11. In a sports meet, one sportsman told another: "There are nine fewer of us here than twice the product of our total number ?" How many sportsman were there at the Meet ?

12. The square of two consecutive positive integers differs by 2003, What is the sum of these two integers ?

13. Given x, y are positive prime numbers. If $x^2 - 2y^2 = 1$, find y.

14. Solve for real values of x, y given that $2x^2 + y^2 + 2xy - 4x + 4 = 0$.

15. If (a + b + c) = 0 then find $\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}$.

16. A = {a, b, c, d, e} is a set of five integers. We take two out of the numbers in A and add. The following ten sums are obtained 0, 6, 11, 12, 17, 20, 23, 26, 32, 37. Find the five integers in the set A.

17. Find the sum of all the coefficients of the polynomial $(x - 2002)^3 (x - 2001)^2 (x - 2000) (x - 3)^3 (x - 2)^2 (x - 1)$.

18. Prove the algebraic identity, $a^3 - b^3 = \left\{\frac{a(a^3-2b^3)}{a^3+b^3}\right\}^3 + \left\{\frac{b(2a^3-b^3)}{a^3+b^3}\right\}^3$.

19. If $\frac{(a-b)(c-d)}{(b-c)(d-a)} = \frac{2012}{2013}$, find the value of $\frac{(a-c)(b-d)}{(a-b)(c-d)}$.

20. Take any natural number. Multiply it with the next two natural numbers. Take another natural number different from the first and do the same as before. Subtract one result from the other to get a positive difference of the original numbers. Add to the quotient the product of the original numbers. prove that the final result is the product of some number by the number next above it.

21. Find all the positive integral solution of the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{2013}$.

22. If a, b, c, d > 0 and $a^4 + b^4 + c^4 + d^4 = 4abcd$, prove that a = b = c = d.

23. Given a + b = c + d and $a^3 + b^3 = c^3 + d^3$. Prove that $a^{2009} + b^{2009} = c^{2009} + d^{2009}$.

24. Given $x + y = \frac{5}{2}, x^2 + y^2 = \frac{13}{4}$, find the value of $x^5 + y^5$.

25. If a, b, c are positive constants, solve the equation $\frac{x-a-b}{c} + \frac{x-b-c}{a} + \frac{x-c-a}{b} = 3$. Prove that x = a + b + c.

26. Find the number of real solutions of the equation $1 + x + x^2 + x^3 = x^4 + x^5$.

27. Find the remainder when x^{100} is divided by $x^2 - 3x + 2$.

28. Given that the equation ax + 4 = 3x – b has more than 1 solution for x. Find the value of $(4a + 3b)^{2015}$.

29. The numbers a, b, c are the digits of a three digit number which satisfy 49a + 7b + c = 286. What is the value of 100 a + 10b + c.

30. A hare, pursued by a gray-hound, is 50 of her own leaps ahead of him. In the time hare takes 4 leaps, the gray-hound takes 3 leaps. In one leap the hare goes $1\frac{3}{4}$ meter and her gray-hound $2\frac{3}{4}$ meter. In how many leaps will the gray-hound overtake the hare ?

31. If $\sqrt{a-x}+\sqrt{b-x}+\sqrt{c-x}=0,$ show that (a + b + c + 3x) (a + b + c – x) = 4(ab + bc + ca)

32. Some amount of work has to be completed. Anand, Bilal and Charles offered to do the job. Anand would alone take a times as many days as Bilal and Charles working together. Bilal would alone take b times as many days as Anand and Charles together. Charles would alone take c times as many days as Anand and Bilal together, Show that $\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=2$

33. Find the value of a,b,c which will make each of the expressions $x^4 + ax^3 + bx^2 + cx + 1$ and $x^4 + 2ax^3 + 2bx^2 + 2cx + 1$ a perfect square.

34. Find a, b, c if they are real numbers a+ b = 4 and $2c^2 - ab = 4\sqrt{3}\,c-10$.

35. When $a = 2^{2014}$ and $b = 2^{2015}$, prove that $\left\{\frac{\frac{(a+b)^2+(a-b)^2}{b-a}-(a+b)}{\frac{1}{b-a}-\frac{1}{a+b}}\right\}\div\left\{\frac{(a+b)^3+(b-a)^3}{(a+b)^2-(a-b)^2}\right\}$ is divisible by 3.

36. If $(x + y + z)^3 = (y + z - x)^3 + (z + x - y)^3 + (x + y - z)^3 + kxyz$ find the numerical value of k. If a = 2015, b = 2014, $c = \frac{1}{2014}$, prove that

$(a + b + c)^3 - (a + b - c)^3 - (b + c - a)^3 - (c + a - b)^3 - 23abc = 2015$

37. If $xy = ab(a + b)$ and $x^2 + y^2 - xy = a^3 + b^3$ find the value of $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right)$

38. (a) ABC is a triangle in which AB = 24, BC = 10 and CA = 26. P is a point inside the triangle. Perpendiculars are drawn to BC, AB and AC. Length of these perpendiculars respectively are x, y and z. Find the numerical value of 5x + 12y + 13z.

(b) If

$$x^2(y + z) = a^2,$$

$$y^2(z + x) = b^2,$$

$$z^2(x + y) = c^2,$$

$$xyz = abc$$

prove that $a^2 + b^2 + c^2 + 2abc = 1$

39. If

$$X = \frac{a^2-(2b-3c)^2}{(3c+a)^2-4b^2}+\frac{4b^2-(3c-a)^2}{(a+2b)^2-9c^2}+\frac{9c^2-(a-2b)^2}{(2b+3c)^2-a^2}$$

$$Y = \frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}$$

find 2017(X + Y).

40. If a, b, c are positive real number such that no two of them are equal, show that $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)$ is always positive.

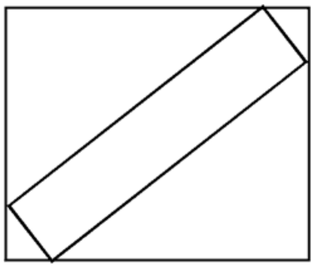
ANSWERS

1. 50 **2.** Number of cows = 7, number of chickens = any whole number				**3.** 20, 30, 5, 625	**4.** 21, 42, 63, 84
5. 2 hrs. 24 min.	**6.** 12 kms	**7.** Rs. 48	**8.** Rs.12	**10.** 19	**11.** 47
12. 2003	**13.** y = 2	**14.** x = 2, y = – 2	**15.** 4	**16.** A{–3,3,9,14,23}	
17. Sum = 0	**19.** $-\frac{1}{2012}$	**21.** 27 solutions	**24.** $\frac{275}{32}$	**26.** 3	
27. $(2^{100}-1)x+(2-2^{100})$		**28.** 0	**29.** 556	**30.** 210	
33. (2, 3, 2) (–2, 3, –2)					
34. a = b = 2; c = $\sqrt{3}$		**36.** k = 24	**37.** 0	**38.** 120	**39.** 4034

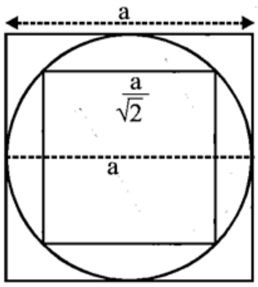
CHAPTER 4 Geometry

1. A rectangle has sides of integer lengths (in cm) and an area of 36 cm^2. What is the maximum possible perimeter of the rectangle?

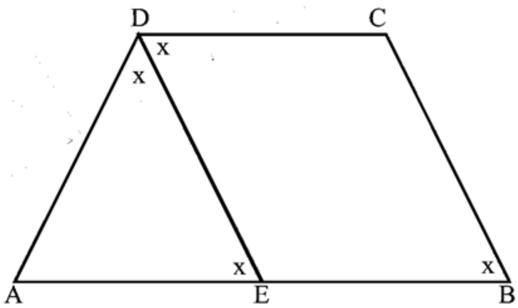
2. Identical isosceles right triangles are removed from opposite corners of a square resulting in a rectangle. If the sum of the areas of the cut-off pieces is 450, find the length of the diagonal of the rectangle.



3. From a square metal plate, a circle of maximum size is cut out; again from this circular plate a square of maximum size is cut out. Find the ratio of the metal wasted to the metal of the original square.

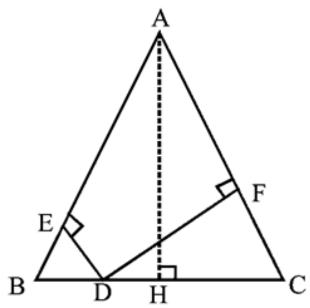


4. In a trapezium ABCD, AB||CD and $\angle D = 2\angle B$. If DC = p and AD = q find AB.



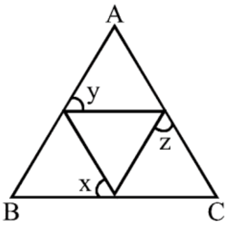
5. The sum of the lengths of the three sides of a right triangle is 18. The sum of the squares of the lengths of the three sides is 128. Find the area of the triangle.

6. ABC is an equilateral triangle (see figure). D is some point on BC. If DE = 3 and DF = 7 find the length of altitude form A to BC.



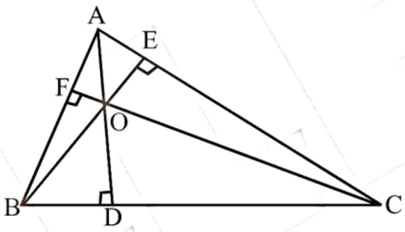
7. In triangle ABC, we are given that $\angle A = 90°$. Median AM, angle bisector AK and the altitude AH are drawn. Prove that $\angle MAK = \angle KAH$.

8. An equilateral triangle is drawn inside an isosceles triangle, as shown in the figure. Show that x is the arithmetic mean of y and z.

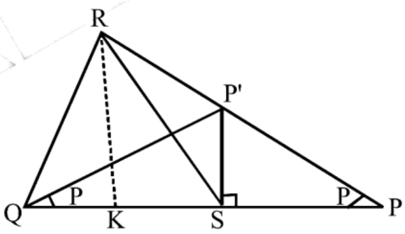


9. Consider the collection C of all isosceles triangles of area 48 sq. units, whose base and heights are integers. How many triangles are there in C? How many triangles in C have their equal sides also of integral lengths?

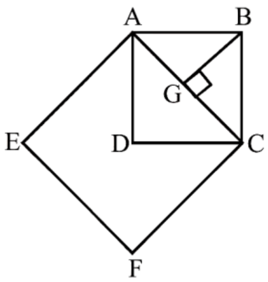
10. Let ABC be an acute angled triangle with AD, BE, CF as the altitudes (i.e., D is the foot of the pependicular from A on BC and so on...). If the altitudes meet at the point O, find the measure of the angles $\angle BOC$, $\angle COA$, $\angle AOB$ in terms of the angles $\angle A$, $\angle B$, $\angle C$ of the triangle ABC.



11. In a triangle PQR, S is the mid point of PQ and PR > QR. Prove that PSR is obtuse.



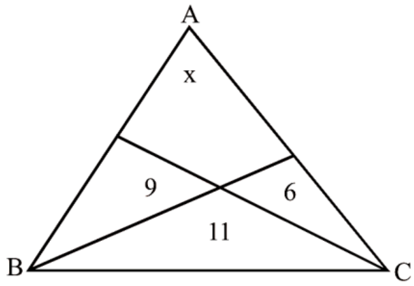
12. In the figure ABCD and AEFC are squares. ΔABG is a right triangle. If ΔABG has an area $\frac{1}{4}$, find the area of the polygon AEFCD.



13. The areas of the six sides of a closed rectangular box are (in cm^2) 48, 80, 60, 48, 80, and 60. Find its volume.

14. 42 identical cubes, each with 1cm edge are glued together to form a cuboid. If the perimeter of the base of the cuboid is 18 cm. Find the height of the cuboid.

15. Triangle ABC is divided into four regions with areas as shown in the diagram. Find x.

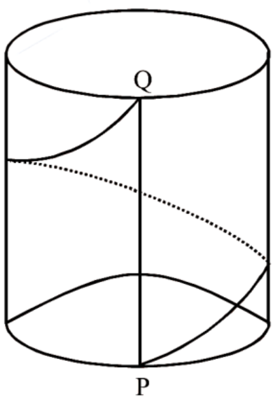


16. A circle is inscribled in a rhombus, one of whose angles is 60°. Find the ratio of area of the rhombus to the area of the inscribed circle.

17. ABCD is a cyclic quadrilateral (which means that a circle passes through the vertices A,B,C,D). In other words the vertices A,B,C,D, in that order, lie on a circle. If the diagonals AC and BD cut at right angles at E, prove that $AE^2 + BE^2 + CE^2 + DE^2 = 4R^2$ where R is the radius of the circle ABCD.

18. About how many lines can one rotate a regular hexagon through some angle $x°(0° < x° < 360°)$, so that the hexagon returns to its original position?

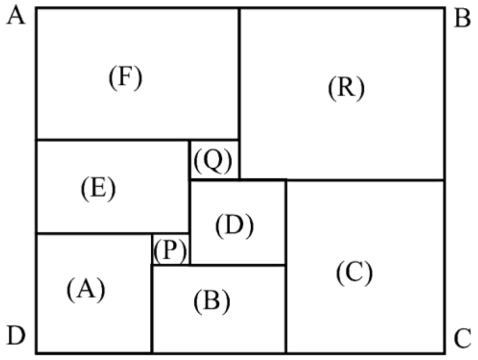
19. Consider a cylinder of height 4cm and the perimeter of the base circle 3cm. P is a point on the lower rim and Q is the vertically point above P on the upper rim. A thread is wound once round the cylinder starting at P and ending at Q what is the length of the thread?



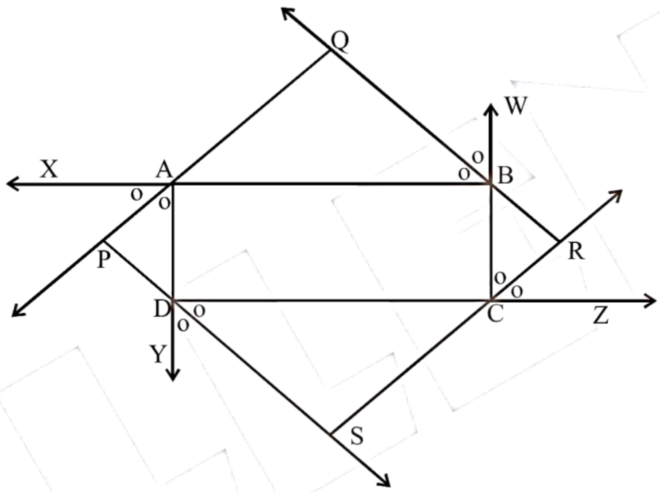
20. A child has at its disposal n small wooden cubes, all the same size. With them he tries to build the largest cube he can, but discovers that he is short by exactly one single row of small cubes that would have formed an edge of the large cube. Prove that n is a multiple of 6.

21. Given that $a^2 - b^2 = 105$ and a and b are two relatively prime positive integers (two positive integers m and n are relatively prime if their g.c.d. (m,n) =1), find all such a and b. After having found all such a and b, if one draws a triangle ABC with sides having lengths $a^2 - b^2$, $a^2 + b^2$ and 2ab, find the area of all such triangles.

22. Nine squares are arranged to form a rectangle ABCD. The smallest square P has an area 4 sq. units. Find the areas of Q and R.



23. ABCD is a rectangle. The sides are extended and the external angles are bisected and the bisectors are produced in both ways to form a quadrilateral. Prove that the quadrilateral is a square.



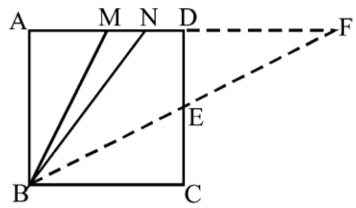
24. If a finite straight line segment is divided into two parts so that the rectangle contained by the whole and first part is equal to the square on the other part, prove that the square described on one of the diagonals of the rectangle contained by the whole and the first part is three times the square on the other part.

25. ABCD is a square. The diagonals AC, BD, cut at E. From B a perpendicular is drawn to the bisector of $\angle DCA$ and it cuts AC at P and DC at Q. Prove that DQ = 2PE.

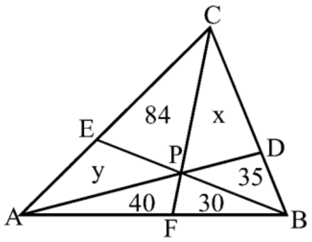
26. In an isosceles triangle ABC, AC = BC, $\angle BAC$ is bisected by AD where D lies on BC. It is found that AD = AB. Then, find $\angle ACB$.

27. The lengths of the sides of a triangle are integers. The length of the longest side is 11. How many such triangles are there ?

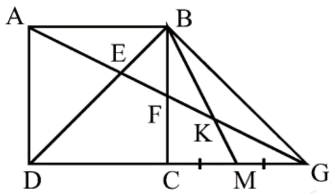
28. In square ABCD, M is the midpoint of AD and N is the midpoint of MD. Prove that $\angle NBC = 2\angle ABM$.



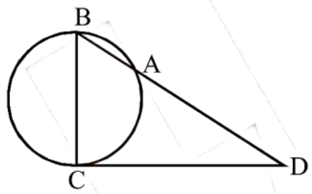
29. As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



30. In the diagram below,ABCDis a square, side DC is produced to a point G so that CG = DC, and M is the midpoint of CG. The line AG meets BD, BC and BM at E, F and K, respectively. Find the ratio AE : EF : FK.



31. In the figure below, A, B and C are points on the circle, CD is tangent to the circle at C, BC is a diameter of the circle and BD cuts the circle at A. If AB = 5 cm and AD = 4 cm, find CD.



32. A convex pentagon ABCDE has the following property: the five triangles ABC, BCD, CDE, DEA, EAB have same area 1. Prove that all such pentagons have an equal area, and there are infinitely many distinct such pentagons.

33. Given that in a right triangle the length of a leg of the right angle is 11 and the lengths of the other two sides are both positive integers. Find the perimeter of the triangle.

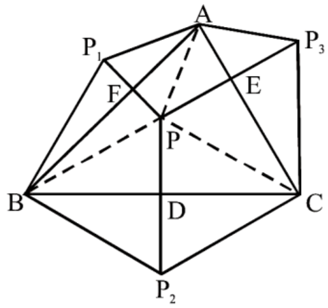
34. What is the positive integer n such that $n + 2$, $4n$ and $5n - 2$ are the side lengths of a right triangle and $5n - 2 > 4n > n + 2$?

35. The perimeter of a triangle is 20. The lengths of three sides are all integers.

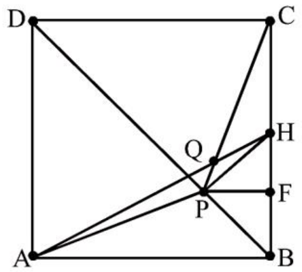
How many triangle are there are not congruent ?

36. ABC is a triangle, with $\angle ABC = \angle ACB$, BD, DE and EF bisect angles $\angle ABC$, $\angle BDC$ and $\angle DEC$ respectively. EF = EC. Calculate the angles of the triangle ABC.

37. In the diagram given below, P is an interior point of ΔABC, $PP_1 \perp AB$, $PP_2 \perp BC$, $PP_3 \perp AC$, and $BP_1 = BP_2$, $CP_2 = CP_3$, prove that $AP_1 = AP_3$.



38. Let P be any point on the diagonal BD of a rectangle ABCD. F is the foot of the perpendicular from P to BC. H is point on the side BC such that FB=FH. PC cuts AH in Q. Show that Area of ΔAPQ = Area of ΔCHQ.



39. Q, R are the midpoints of the sides AC, AB of the isosceles triangle ABC in which AB = AC. The median AD is produced to E so that DE = AD. EQ and ER are joined to cut BC in N and M respectively. Show that AMEN is a rhombus.

40. ABCD is a square. The diagonals AC, BD, cut at E. From B a perpendicular is drawn to the bisector of ∠DCA and it cuts AC at P and DC at Q. Prove that DQ = 2PE.

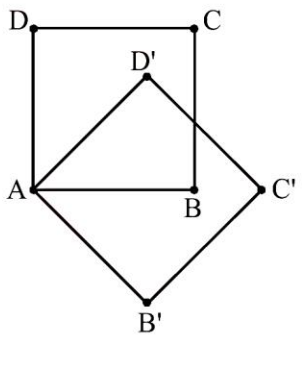
41. Prove that the feet of the perpendiculars drawn from the vertices of a parallelogram onto its diagonals are the vertices of another paralleogram.

42. ABC is an acute angled triangle. P, Q are points on AB and AC respectively such that the area of ΔAPC = area of ΔAQB. A line is drawn through B parallel to AC and meets the line through Q parallel AB at S. QS cuts BC at R. Prove that RS = AP.

43. A point P is taken within a rhombus ABCD such that PA = PC. Show that B, P, D are collinear.

44. In a triangle ABC, ∠C = 90° and BC = 3AC. Points D, E lie on CB such that CD = DE = EB. Prove that ∠ABC + ∠AEC + ∠ADC = 90°

45. The square ABCD of side length 'a' cm is rotated about A in the clockwise direction by an angle 45° to become the square AB'C'D'. Show that the shaded area is $\left(\sqrt{2}-1\right)a^2$ square cms.



ANSWERS

1. 74 cm	**2.** 30	**3.** 1 : 2	**4.** AB = p + q	**5.** 9	**6.** 10
9. 12, 2	**10.** $\angle BOC = \angle B + \angle C, \angle COA = \angle C + \angle A, \angle AOB = \angle A + \angle B$			**12.** 3/2	**13.** 480 (cm^3)
14. 3 cm	**15.** $\frac{1998}{67}$	**16.** $8 : \pi\sqrt{3}$	**18.** 7	**19.** 5 cm	
21. 289380, 31920, 10920, 4620		**22.** Area of Q = 64 sq. uints, Area of R = 1296 sq. units			**26.** 36°
27. 36	**29.** 315	**30.** 2:1:1	**31.** 6	**32.** $\frac{5+\sqrt{5}}{2}$	**33.** 132
34. 3	**35.** 8	**36.** 36°			

CHAPTER 5 Number Theory

1. Show that if n is a positive integer, then n^3-n is always divisible by 3.

2. Show that among all the prime numbers there is no greatest prime.

3. Find the smallest number which when divided by 10 leaves a remainder of 9, when divided by 9 leaves a remainder of 8, when divided by 8 leaves a remainder of 7,... etc. down to where, when divided by 2 it leaves a remainder 1.

4. The sum of two natural number is 100. Show that their product cannot be greater than 2500.

5. The sum of two numbers is a constant. Prove that the product is the maximum when the number are equal.

6. A is a set of 2004 positive integers. Show that there is a pair of elements in A whose difference is divisible by 2003.

7. $a > b > c$ are three consecutive numbers. Show that $a^3 \neq b^3 + c^3$.

8. Is the statement "If p and $p^2 + 2$ are primes then $p^3 + 2$ is also a prime" true or false? Give reasons for your answer.

9. Find the natural number n such that $2^{13}+2^{10}+2^n$ is a perfect square of an integer.

10. If the square of any odd natural number is divided by 8, show that the remainder will always be 1.

11. A number when added to either 100 or to 168 yields a perfect square natural number. What are these square number?

12. Find the first positive integer whose square ends in three 4's.

13. Let P denote the product of first n prime numbers (with $n > 2$). For what values of n we have.

1. P - 1 is a perfect square

2. P + 1 is a perfect square

14. Find all three digit and four digit natural numbers such that the product of the digits is a prime number. Find the sum of all such three digit numbers and the sum of all such four digit numbers. Find the biggest prime factor of each sum.

15. Let a sequence of numbers be denoted as t_1, t_2, t_3, ..., where $t_1 = 1$ and $t_n = t_{n-1} + n$. (n is a natural number). Find t_2, t_3, t_4, t_{10}, t_{2011}.

16. When written out completely 16^{2011} has m digits and 625^{2011} has n digits. Find the value of (m + n).

17. Four digit numbers are formed by four different digits a, b, c, d (none of them is zero) without any repetition of digits. Prove that when the sum of all such numbers when divided by the sum of the digits a, b, c, d, the quotient is 6666.

18. Falguni puts 12 plastic bags inside another plastic bag. Each of these 12 bags is either empty or contains 12 other plastic bags. All together if 12 bags were non-empty, find the total number of bags.

19. A, B and C are the digits of a three digit number ABC (A, C $\neq$ 0). The number got by reversing the digits (also a three digit number) is added to ABC and the sum is found to be a square number. Find all such three digit numbers.

20. Find the number of numbers coprime to and less than 2012. Find their sum. Find also the quotient when this sum is divided by 2012.

21. Composite twins are defined below:

(a) Odd composite twin : let a and a + 2 be two odd composite numbers. If (a – 2) and (a + 4) are primes then (a, a + 2) is called an "odd composite twin".

(b) Even composite twin : Let b and b + 2 two even numbers (b > 2). If (b – 1) and (b + 3) are primes then (b, b + 2) is called an even composite twin.

List all composite twins less than or equal to 100.

22. (a) A single digit natural number is increased by 10. The obtained number is now increased by the same percentage as in the first increase. The result is 72. Find the original single digit number.

(b) After two price reductions by one and the same percent the price of an article is reduced from Rs. 250 to Rs. 160. By how much percent was the price reduced each time. Write detailed steps.

23. abcde is a five digit number. Show that abcde is divisible by 7 if and only if the number abcd – (2 × e) is divisible by 7.

24. A three digit number with digits A, B, C in that order is divisible by 9. A is an odd digit and C is an even digit. B and C are non zero. Find the number of such three digit numbers.

25. Simplify

$$P = \frac{1}{2\sqrt{1}+\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \ldots + \frac{1}{100\sqrt{99}+99\sqrt{100}}.$$

26. Find the smallest positive integer k such that $2^{69} + k$ is divisible by 127.

27. A positive integer is called a "good number" if it is equal to four times of the sum of its digits. Find the sum of all good numbers.

28. Find the smallest natural number n which has the following properties:

(a) Its decimal representation has 6 as the last digit.

(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

29. Find all the three digit number n satisfying the condition that if 3 is added, the sum of digits of the resultant number is 1/3 of that of n.

30. Find the number of the pairs (x, y) of two positive integers, such that N = 23x + 92y is a perfect square number less than or equal to 2392.

31. Find all the natural number n such that $n^2 - 19n + 91$ is a perfect square.

32. A three digit number in base 7 when expressed in base 9 has its digits reversed in order. Find the number in base 7 and base 10.

33. (a) Two regular polygons have the number of their sides in the ratio 3 : 2 and the interior angles in the ratio 10 : 9. Find the number of sides of the polygons.

(b) Find two natural numbers such that their difference, sum and the product is to one another as 1,7 and 24.

ANSWERS

3. N = 2519	**8.** True	**9.** 14	**11.** 324, 256	**12.** $38^2 = 1444$
14. 101	**15.** $t_1 = 1, t_2 = 3, t_3 = 6, t_4 = 10, t_{10} = 55, t_{2011} = 2023066$			**16.** 8045
18. 145	**19.** 143, 242, 341, 164, 263, 362, 461, 198, 297, 396, 495, 594, 693, 792, 891			
20. sum = 1010024, quotient = 502		**22.** (a) 2, (b) 20%	**24.** 20	**25.** $\frac{9}{10}$
26. 63	**27.** 120	**28.** 153846	**29.** 108, 117, 207	**30.** 27
31. 9 or 10	**32.** 248,503	**33.** (a) 12, 8, (b) 6, 8		